Resonant control of the Rössler system

Valery Tereshko[1,*] and Elena Shchekinova[2,†]
[1]*Communications Research Laboratory, McMaster University, 1280 Main Street West, Hamilton, Ontario Canada L8S 4K1 and The Nonlinear Centre, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom*
[2]*Center for Dynamical Systems and Nonlinear Studies, School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332*

(Received 3 December 1997)

We develop a method of control, ''resonant control,'' when a weak resonant perturbation is tuned so as to drive the system into naturally occurring regimes, namely, periodic orbits, which happen to be unstable for some nominal parameter value. The results show that nonfeedback control by periodic perturbations can be goal oriented, and a final state can be predictably targeted. The method allows us to alter nonchaotic as well as chaotic dynamics using only small perturbations.

[S1063-651X(98)11507-6]

PACS number(s): 05.45.+b

The problem of dynamical system control consists of a goal-oriented alteration of its dynamics. For example, in many cases it is important not only to suppress a chaotic behavior but to convert it into a desired regular one. A feedback method for stabilization of unstable periodic orbits embedded in the chaotic attractor was developed for that [1]. Its advantage is the use of weak perturbations. Indeed, because of the ergodicity of chaotic systems, sooner or later the trajectory will fall into the vicinity of the desired unstable orbit. Then it is sufficient only to move the stable manifold of the corresponding unstable fixed point in the Poincaré section to the system state point to stabilize the former. In cases when the reciprocal of the maximal Lyapunov exponent is short compared to the time between perturbations, that can lead to occasional bursts of lost control, or when the dynamics is too fast for real-time computation of the control signal, continuous control strategies based on delayed self-controlling feedback, or a combination of feedback with a periodic perturbation, can be applied [2]. These methods (and their modifications [3]) have been experimentally verified for different chaotic systems [4]. However, in the case when the system state is not immediately accessible, the only way of control is the use of nonfeedback techniques.

One of the approaches to nonfeedback control is the nonlinear entrainment method [5]. It requires a knowledge of the system equations to construct control forces which, however, can have a large amplitude and a complicated shape, and the basins of entrainment can have a very complicated structure. Typically, this method can require as many control forces as there are dimensions of the system.

In contrast, there are many examples of converting chaos to a periodic motion by exposing a system to only one periodic force or parameter modulation [6–14]. Provided that a small perturbation is applied, the controlled periodic orbit closely traces the corresponding unperturbed one.

Although periodic perturbation methods are easy to realize in practice, their main deficiency is that often it is difficult to anticipate the result of perturbation, which can give rise to an undesired behavior of system. In other words, there is no common concept for construction of appropriate perturbations which direct the trajectory to the target.

In this paper we propose a method of nonfeedback control when the perturbation is tuned so as to goal-orientedly drive the system into naturally occurring regimes, namely, periodic orbits, which happen to be unstable for some nominal parameter value. The amplitude of this perturbation is very small, but its resonant effect is enough to alter the system dynamics drastically. We call this type of control *resonant control*.

Let us formulate the conditions of resonant control for regular dynamics. The generalization to the chaotic case is straightforward. The main idea of resonant control is that the wave form of the perturbation must be tailored to suit the wave forms of both the controlled variable and the desired response. This means that (i) the period, phase, amplitude, and shape of perturbation have to be tuned so as to compensate for the difference between current and desired wave forms of the controlled variable to provide goal-oriented targeting; and (ii) the symmetry of the perturbation must correspond to the symmetry of the desired wave form so as not to destabilize the latter. Provided that the period of the desired response is a multiple of the period of the current wave form, condition (i) guarantees the perturbation to be resonant, i.e., its period has to be equal to or a multiple of the period of current cycle.

The concept of resonant control can be considered as a generalization of the original concept of geometrical resonance [7], requiring that the control force preserve a natural response from the underlying conservative system, to the case when the weak perturbation drives the trajectory to follow an *a priori* chosen natural response from the unperturbed dissipative system.

*Author to whom correspondence should be addressed. Present address: The Nonlinear Centre, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K. Electronic address: V.Tereshko@damtp.cam.ac.uk
†Electronic address: elenash@math.gatech.edu

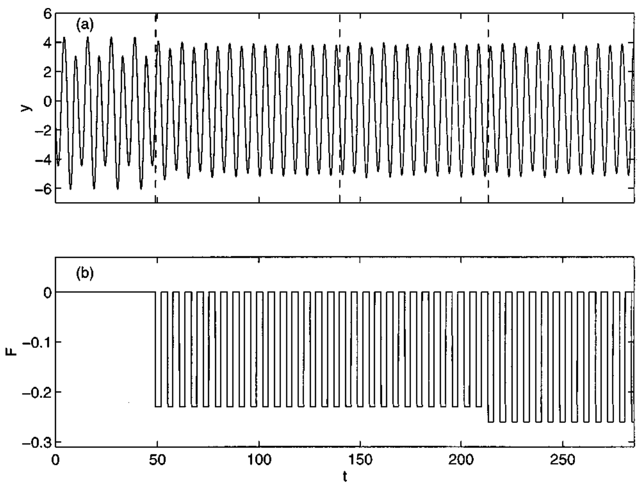

FIG. 1. (a) Control of the period-2 cycle ($c=3.2$) onto the period-1 cycle. Beginning with $t=140$ and 213.4, random fluctuating forces with mean value 0, mean-squared value 1, and amplitudes 0.15 and 0.25, respectively act on the system. (b) Control perturbation with period $T_1=5.8679$ and amplitudes $\alpha=0.23$ and 0.26, turned on at $t=49.1$ and 213.4, respectively.

Let us consider controlling the Rössler system [15] as an example:

$$\dot{x}=-y-z,$$
$$\dot{y}=x+ay+F(t), \tag{1}$$
$$\dot{z}=b+z(x-c),$$

where $a=b=0.2$, c is the bifurcation parameter, and $F(t)$ is the control force.

Let the bifurcation parameter correspond to a stable period-2 cycle ($c=3.2$). To stabilize the cycle of period 1, we use the negative pulse train, shown in Fig. 1(b), with period $T_1=2\pi/\omega_1$, where $\omega_1=1.0708$ is the fundamental frequency of the Rössler system [16]. If these pulses are in phase with the positive peaks of the controlled variable y, then the perturbation will target the trajectory to the period-1 cycle, so that, at a sufficient amplitude of force, it will be stabilized [see Fig. 1(a)]. Beginning with $t=140$ and 213.4, random fluctuating forces with mean value 0, mean-squared value 1, and amplitudes 0.15, and 0.25, respectively, act on the right-hand sides of Eqs. (1). To control the system with a higher level of noise effectively, we increased the amplitude of perturbation [see Fig. 3(b), $t\geqslant 213.4$]. The results show that a resonant perturbation controls the noisy system rather successfully, provided that its amplitude is not less than the amplitude of the random forces.

Consider the opposite transition. Let the system dynamics correspond to the period-1 cycle ($c=2.5$). In order to direct the trajectory to the period-2 cycle it is necessary to break the symmetry of the former cycle. For that, the system is forced so that every first and second peak of the controlled variable y will be pushed up and down, respectively. So, we use a train of positive and negative pulses with period $T_2=11.4688$, which corresponds to the first subharmonic of the power spectrum at the given value of parameter c [16] [see Fig. 2(b), $48.7\leqslant t<140.4$]. If the pulses are phased with positive (or negative) peaks of the controlled variable, then at sufficient amplitudes of perturbation, the cycle of period 2 is stabilized [see Fig. 2(a)]. Keeping the same phase and using the pulse trains with period $T_4=2T_2$ and $T_8=4T_2$, as it is shown in Fig. 2(b) in the time intervals $140.4\leqslant t<232.3$ and $t\geqslant 232.3$, one stabilizes cycles of periods 4 and 8, respectively [see Fig. 2(a)].

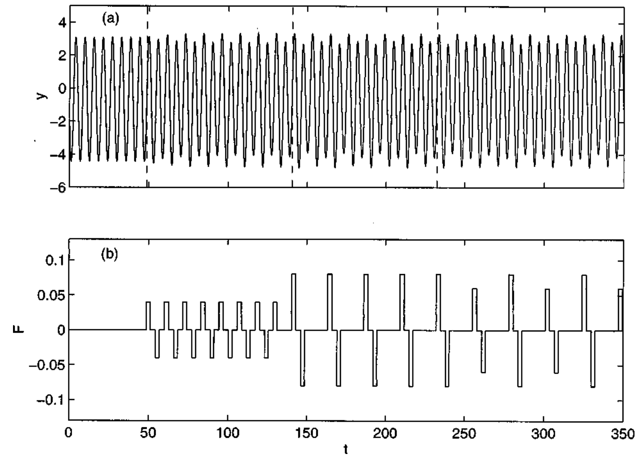

FIG. 2. (a) Control of the period-1 cycle ($c=2.5$) onto the period-2, -4, and -8 cycles, respectively. (b) Control perturbation with period $T_2=11.4688$, $T_4=2T_2$, and $T_8=4T_2$, and amplitudes $\alpha=0.04$, 0.08, and 0.08 (0.06 for lower pulses) turned on at $t=48.7$, 140.4, and 232.3, respectively.

The phase of perturbation can play a crucial role in resonant control. Keeping a perturbation precisely in phase with the controlled variable provides the most effective way of targeting, that ensures the smallest sufficient amplitude of external force. However, changing the phase of perturbation on the opposite one (by 180°) can bring a system to an ''opposite'' behavior. Let the period-4 cycle be stable for the unperturbed system ($c=4$). To stabilize the period-8 cycle, we use a perturbation with period $T_2=11.5729$, as shown in Fig. 3(b). The symmetry of this perturbation fits well not only a period-2 cycle but higher order cycles too. Indeed, fit-

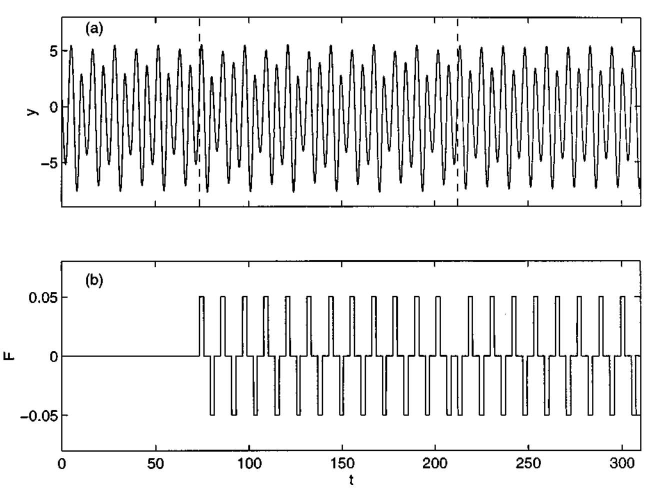

FIG. 3. (a) Control of the period-4 cycle ($c=4$) onto the period-8 and -2 cycles. (b) Control perturbation with period $T_2=11.5729$ and amplitude $\alpha=0.05$ turned on at $t=73.5$. The phase is changed on 180° at $t=212.4$.

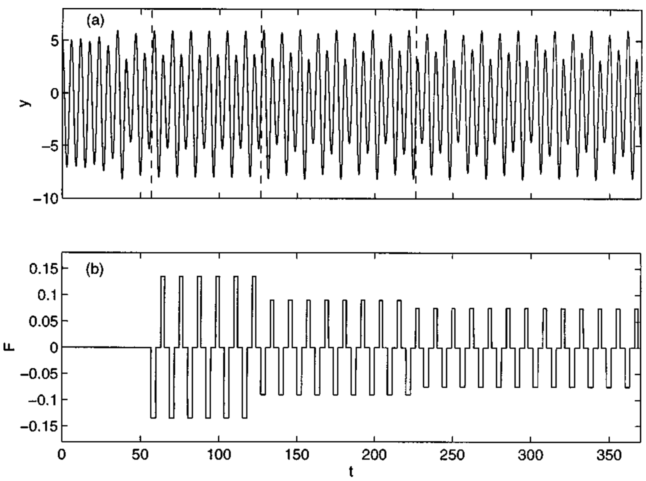

FIG. 4. (a) Control of the chaotic attractor ($c=4.45$) onto the period-2, -4, and -8 cycles. (b) Control perturbation with period $T_2=11.6667$ and amplitudes $\alpha=0.135$, 0.09, and 0.075 turned on at $t=56.8$, 126.8, and 226.3, respectively.

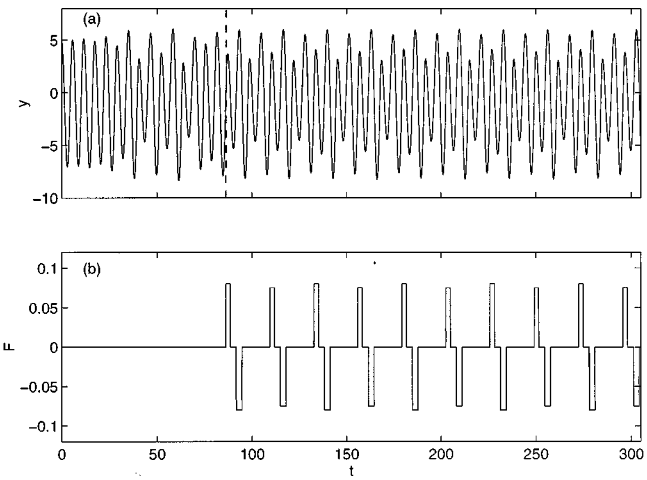

FIG. 5. (a) Control of the chaotic attractor ($c=4.45$) onto the period-8 cycle. (b) Control perturbation with period $T_8=46.5778$ and amplitude $\alpha=0.08$ (0.075 for lower pulses) turned on at $t=86.3$.

ting negative and positive pulses to low and high positive peaks of variable y, respectively, we will pull the former down and the latter up, thus directing the trajectory to the desired cycle [see Fig. 3(a), $73.5\leqslant t<212.4$]. Let us change the phase of the pulse train on 180°, as shown in Fig. 3(b) at $t=212.4$. Unlike the previous case, this perturbation directs the trajectory to the period-2 cycle [see Fig. 3(a), $t\geqslant 212.4$]. So the same perturbation but with different phases provides entirely different dynamics.

The same approach can be applied to chaotic dynamics, when the system trajectory is in the vicinity of the cycle to be stabilized. Let the system dynamics correspond to a chaotic attractor ($c=4.45$). Choosing the phase of perturbation, say, a pulse train with period $T_2=11.6667$, to fit the positive (or negative) peaks of variable y, one stabilizes the period-2 cycle (see Fig. 4, $56.8\leqslant t<126.8$). As the symmetry of this perturbation fits the symmetry of higher order cycles, it is possible to stabilize the latter. Indeed, decreasing the amplitude of forcing, one can control the cycles of higher periods down to the threshold value when the dynamics becomes chaotic again. Figure 4 shows the controlled cycle of periods 4 and 8 in the time interval $126.8\leqslant t<226.3$ and $t\geqslant 226.3$, respectively. So, one obtains an inverse cascade of period-doubling bifurcations, the amplitude of perturbation varying much less than the bifurcation parameter c in the straight period-doubling cascade. Similar results are obtained by using the pulse train with period $T_4=2T_2$ as a control perturbation. This perturbation stabilizes the period-4 cycle and higher order cycles which are multiples of it.

Another way to stabilize higher order cycles is to use a perturbation with the same period as the period of the cycle to be controlled. Figure 5 illustrates the stabilization of a period-8 cycle by the pulse train with period $T_8=46.5778$.

Tuning the control force to the wave form of variable x or z, one can obtain the same results. However, the perturbations which are tailored to variable y need slightly smaller amplitudes to achieve stabilization, probably because the system trajectory and trajectories of unstable cycles are closer in the phase space in the direction of axis y. The same results as above can also be obtained by a tiny modulation of parameter c, provided that this modulation fits the wave form of one of the system variables.

Along with the period and phase, the shape of the perturbation can be very important too. Indeed, reshaping a perturbation can lead to a redistribution of energy between its harmonics, and, hence, to an enhancement or suppression of the harmonics which govern the system dynamics. The example of a forced Duffing equation, when reshaping the driving force increases the effective amplitude of a harmonic which is responsible for a transition from regular to chaotic dynamics, is considered in Ref. [8].

In contrast to the nonfeedback scheme for creation of desired outputs outlined in Ref. [9], our method does not need *a priori* analytical knowledge of the system dynamics. The fundamental frequency can be obtained from the power spectrum. The procedure does not use any specific information about the desired response. The perturbations have been chosen so as to target the trajectory to anticipated unstable periodic orbits. To reveal the unstable skeleton of the system in advance, conventional techniques may be applied. If the system analysis (say, power spectrum, Poincaré section, etc.) indicates a chaotic dynamics, then unstable periodic orbits can be extracted from a delay coordinate vector of an observable variable [17]. Otherwise, if the system possesses regular dynamics, it is possible to utilize noise to learn about unstable orbits [18].

The proposed approach can explain some previous studies on chaos suppression. For example, applied to the Rössler system, the method of chaos suppression through changes in the system variables in the Poincaré and Lorenz sections [10] is nothing but directing the system trajectory by resonant periodic pulses. Another example, when weak periodic modulation of a bifurcation parameter of the Rössler system tames chaos, is given in Ref. [11]. Again, the perturbation appears to fit both the system variable and the stabilized cycle. The importance of choosing an appropriate phase of perturbation for eliminating chaos was shown analytically [7,12] and numerically [7,12,13] for a forced Duffing equation, and experimentally for a laser with modulated losses [14].

The goal of the present paper is to show that (i) nonfeedback control by periodic perturbations can be goal oriented, and a final state can be predictably targeted; and (ii) nonchaotic dynamics can be altered by weak perturbations. We showed that nonfeedback control by resonant perturbations, ''resonant control,'' works very well for the Rössler system. The application of the procedure to other systems will be considered in a forthcoming paper.

Both of us are grateful to D. Goodings for fruitful discussions, as well as S. Haykin and the Neural Networks Group of the CRL for providing facilities for us to work. V. T. wishes to acknowledge support from the Royal Society, and to thank R. S. MacKay for discussions and helpful comments.

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Nature (London) **363**, 411 (1993).

[2] K. Pyragas, Phys. Lett. A **170**, 421 (1992).

[3] J. F. Lindner and W. L. Ditto, Appl. Mech. Rev. **48**, 795 (1995), and references therein.

[4] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990); K. Pyragas and A. Tamaševičius, Phys. Lett. A **180**, 99 (1993); W. L. Ditto, M. L. Spano, and J. F. Lindner, Physica D **86**, 198 (1995).

[5] A. Hübler and E. Lüscher, Naturwissenschaften **76**, 67 (1989); E. A. Jackson and A. Hübler, Physica D **44**, 407 (1990).

[6] V. V. Alexeev and A. Yu. Loskutov, Dokl. Akad. Nauk **293**, 1346 (1987) [Sov. Phys. Dokl. **32**, 270 (1987)]; R. Lima and M. Pettini, Phys. Rev. A **41**, 726 (1990); A. Azevedo and S. M. Rezende, Phys. Rev. Lett. **66**, 1342 (1991); Y. Braiman and I. Goldhirsch, *ibid.* **66**, 2545 (1991); L. Fronzoni, M. Giocondo, and M. Pettini, Phys. Rev. A **43**, 6483 (1991); M. Salerno, Phys. Rev. B **44**, 2720 (1991); W. X. Ding, H. Q. She, W. Huang, and C. X. Yu, Phys. Rev. Lett. **72**, 96 (1994); Y. S. Kivshar, F. Rödelsperger, and H. Benner, Phys. Rev. E **49**, 319 (1994); Y. Liu and J. R. Leite, Phys. Lett. A **185**, 35 (1994); M. A. Matías and J. Güémez, Phys. Rev. E **54**, 198 (1996).

[7] R. Chacón, Phys. Rev. Lett. **77**, 482 (1996); Phys. Rev. E **54**, 6153 (1996).

[8] R. Chacón and J. Díaz Bejarano, Phys. Rev. Lett. **71**, 3103 (1993); F. Rödelsperger, Y. S. Kivshar, and H. Benner, Phys. Rev. E **51**, 869 (1995).

[9] M.-C. Ho, J.-L. Chern, and D.-P. Wang, Phys. Lett. A **194**, 159 (1994).

[10] J. M. Gutiérrez, A. Iglesias, J. Güémez, and M. A. Matías, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 1351 (1996).

[11] A. S. Mikhailov and A. Yu. Loskutov, *Foundations of Synergetics II: Chaos and Noise*, 2nd ed. (Springer-Verlag, Berlin, 1996), pp. 97 and 98.

[12] R. Chacón, Phys. Rev. E **51**, 761 (1995); **52**, 2330 (1995).

[13] Z. Qu, G. Hu, G. Yang, and G. Qin, Phys. Rev. Lett. **74**, 1736 (1995).

[14] R. Meucci, W. Gadomski, M. Ciofini, and F. T. Arecchi, Phys. Rev. E **49**, R2528 (1994); D. Dangoisse, J.-C. Celet, and P. Glorieux, *ibid.* **56**, 1396 (1997); V. N. Chizhevsky, R. Corbalán, and A. N. Pisarchik, *ibid.* **56**, 1580 (1997).

[15] O. E. Rössler, Phys. Lett. A **57**, 397 (1976).

[16] N. Ohtomo, A. Sumi, Y. Tanaka, K. Tokiwano, and S. Terachi, J. Phys. Soc. Jpn. **65**, 2811 (1996).

[17] D. Auerbach, P. Cvitanović, J.-P. Eckmann, G. H. Gunaratne, and I. Procaccia, Phys. Rev. Lett. **58**, 2387 (1987); A. Belmonte, M. J. Vinson, J. Glazier, G. H. Gunaratne, and B. Kenny, *ibid.* **61**, 539 (1988); D. P. Lathrop and E. J. Kostelich, Phys. Rev. A **40**, 4028 (1989).

[18] D. J. Christini and J. J. Collins, Phys. Rev. E **52**, 5806 (1995); **53**, R49 (1996).